CONNES EMBEDDABILITY OF GRAPH PRODUCTS

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Abstract. We prove that a graph product of separable II_1-factors that each embed into $\mathcal{R}^{\mathcal{U}}$ embeds again into $\mathcal{R}^{\mathcal{U}}$, i.e. the Connes embedding problem is stable under graph products.

Graph products from a group theoretical construction generalizing free products by adding commutation relations that are dictated by a graph. The construction was first considered by Green in her thesis [Gr90] and important examples of graph products arise as right angled Coxeter groups and right angled Artin groups. The formal definition is as follows.

Definition 0.1. Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. We may assume that Γ has no double edges and no loops, i.e. $(v, v) \notin E\Gamma$. For $v \in V\Gamma$ let G_v be discrete group. Let G_Γ be the graph product group which is the discrete group freely generated by $\mathsf{G}_v, v \in V\Gamma$ subject to the relation $sts^{-1}t^{-1} = 1$ whenever $s \in \mathsf{G}_v$ and $t \in \mathsf{G}_w$ with $(v, w) \in E\Gamma$.

Many stability properties of graph products have recently been found by various authors. For example: soficity [CHR12], Haagerup property [AnDr13] (or [CaFi14]), residual finiteness [Gr90], rapid decay [CHR13], linearity [HsWi99] and many other properties, see e.g. [HeMei95], [AnMi11], [Chi12]. Also in [CaFi14] the operator algebraic graph product was defined. In this paper we show that the Connes embedding problem is preserved by graph products.

Previous results concerning stability of the Connes embedding problem for *free* products were shown by Popa [Pop14] and Isono–Houdayer [HoIs15]. In [Dy08] Brown, Dykema and Jung also found the free entropy dimension of free products. Here we take the approach based on Junge [Jun05] and Nou [Nou06] – also proving the result for free products – by explicitly computing mixed moments of matrix models.

Acknowledgements. The author thanks Marius Junge for useful discussions concerning the approach via free Araki-Woods factors.

1. Preliminaries

1.1. **Von Neumann algebras.** For background on von Neumann algebras we refer to Takesaki his books [Tak03]. We may and will assume that every von Neumann algebra is represented on its standard GNS space. Let $(\mathcal{M}_i, \varphi_i)_{i\in I}$ be an indexed set of von Neumann algebras $\mathcal{M}_i$ with faithful normal states φ_i. Let $\mathcal{U}$ be a free ultrafilter on $\mathcal{I}$. Let $\prod_{i,\mathcal{U}} \mathcal{M}_i$ be the Raynaud ultraproduct of von Neumann algebras which can canonically be identified with $(\prod_{i,\mathcal{U}} (\mathcal{M}_i)_*)^*$, see [Ray02] for details. As in [Ray02] we shall write $(x_i)^\bullet$ for a bounded sequence $(x_i)_{i\in I}$ with $x_i \in \mathcal{M}_i$ identified within $\prod_{i,\mathcal{U}} \mathcal{M}_i$. Such sequences form a σ-weakly dense $*$-subalgebra of $\prod_{i,\mathcal{U}} \mathcal{M}_i$. We use analogous notation for an ultraproduct of states. Let e be the support projection of the ultraproduct state $(\varphi_i)^\bullet$. The ultraproduct von Neumann algebra $\prod_{i,\mathcal{U}}[\mathcal{M}_i, \varphi_i]$ is defined as the corner algebra $e(\prod_{i,\mathcal{U}} \mathcal{M}_i)e$. In case each $(\mathcal{M}_i, \varphi_i)$ is the hyperfinite II_1 factor $\mathcal{R}$ equipped with its unique normal faithful tracial state τ we set for the resulting algebra $\mathcal{R}^{\mathcal{U}} := \prod_{i,\mathcal{U}}[\mathcal{R}, \tau]$. We can now state the following conjecture/problem due to A. Connes.

Connes embedding problem: Every separable II_1 factor $\mathcal{M}$ embeds into $\mathcal{R}^{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$.

Date: June 8, 2015.
MC is supported by the grant SFB 878 "*Groups, geometry and actions*".

Here an embedding means that there exists an injective normal $*$-homomorphism $\mathcal{M} \hookrightarrow \mathcal{R}^{\mathcal{U}}$. As the Connes embedding problem is not known to be true or false we shall say that a von Neumann algebra $\mathcal{M}$ is *Connes embeddable* if an embedding into $\mathcal{R}^{\mathcal{U}}$ exists. We refer to [Oza04] for a survey and to [BCV15] for recent examples of Connes embeddable von Neumann algebras.

Let again $(\mathcal{M}_i, \varphi_i)_{i\in I}$ be an indexed set of von Neumann algebras $\mathcal{M}_i$ with faithful normal states φ_i. Assume that each φ_i is tracial. Let $\mathcal{U}$ be a free ultrafilter on I. Let $\mathcal{I}_{\mathcal{U}}$ be the σ-weakly closed ideal in $\prod_{i,\mathcal{U}} \mathcal{M}_i$ generated by all elements $(x_i)^\bullet$ satisfying $\lim_{i,\mathcal{U}} \tau(x_i^* x_i) = 0$. The Ocneanu ultraproduct $\mathcal{M}^{\mathcal{U}}$ is defined as $\left(\prod_{i,\mathcal{U}} \mathcal{M}_i\right)/\mathcal{I}_{\mathcal{U}}$. The Ocneanu ultraproduct $\mathcal{M}^{\mathcal{U}}$ is isomorphic to the ultraproduct $\prod_{i,\mathcal{U}}[\mathcal{M}_i, \varphi_i]$, see [AnHa14, Corollary 3.28].

1.2. **Graph products.** We refer to [Gr90] and [CaFi14] for the following results. Let Γ be a simplicial graph with vertex set $V\Gamma$ and edge set $E\Gamma$. Simplicial means that Γ does not have double edges and for every $v \in V\Gamma$ we have $(v,v) \notin E\Gamma$. We presume Γ is non-oriented so that $(v,w) \in E\Gamma$ whenever $(w,v) \in E\Gamma$. In this paper we shall assume that $V\Gamma$ is countable so that the graph product of separable von Neumann algebras is again separable. For $v \in V\Gamma$ we set $\mathsf{Link}(v) = \{w \in V\Gamma \mid (w,v) \in E\Gamma\}$. We set $\mathsf{Star}(v) = \mathsf{Link}(v) \cup \{v\}$.

A word is a string of vertices and a word $\mathbf{w} = w_1 \dots w_n$ is called *reduced* if the following property holds: if $w_i = w_j$ with $i < j$ then there exists a $i < k < j$ such that $w_k \notin \mathsf{Link}(w_i)$. We let $\mathcal{W}_{\rm red}$ be the reduced words. We say that two words $\mathbf{v}$ and $\mathbf{w}$ are *equivalent* if they are equivalent modulo the equivalence relation generated by the two relations:

$$
\begin{array}{llll}
\mathrm{I} & (v_1, \dots, v_i, v_{i+1}, \dots, v_n) \simeq (v_1, \dots, v_i, v_{i+2}, \dots, v_n) & \text{if} & v_i = v_{i+1}, \\
\mathrm{II} & (v_1, \dots, v_i, v_{i+1}, \dots, v_n) \simeq (v_1, \dots, v_{i+1}, v_i, \dots, v_n) & \text{if} & v_i \in \mathsf{Link}(v_{i+1}).
\end{array} \tag{1.1}
$$

Moreover, we say that two words $\mathbf{v}$ and $\mathbf{w}$ are *type* I *equivalent* if they are equivalent modulo the sub-equivalence relation generated by relation I. We define the notion *type* II *equivalent* in the analogous way.

Every word is equivalent to a reduced word [CaFi14, Lemma 1.3]. Out of every equivalence class of words we may therefore pick a single distinguished reduced word, which we call minimal. We let $\mathcal{W}_{\rm min}$ be the set of minimal words and $\mathcal{W}_{\rm red}$ be the set of reduced words.

Let $\mathsf{M}_v, v \in V\Gamma$ be von Neumann algebras with normal faithful states φ_v. We set M_v° for the kernel of φ_v. We define the graph product von Neumann algebra in an explicit way, see [CaFi14, Section 2]. Let $\mathcal{H}_v$ be the Hilbert space on which M_v acts and let $\xi_v \in \mathcal{H}_v$ be a distinguished unit vector such that $\varphi_v(\,\cdot\,) = \langle \,\cdot\, \xi_v, \xi_v \rangle$. We let $\mathcal{H}_v^\circ$ be the subspace of $\mathcal{H}_v$ consisting of vectors orthogonal to ξ_v. For a word $\mathbf{v} = v_1 \dots v_n$ we set $\mathcal{H}_{\mathbf{v}} = \mathcal{H}_{v_1}^\circ \otimes \dots \otimes \mathcal{H}_{v_n}^\circ$. By Lemma [CaFi14, Lemma 1.3] we see that if $\mathbf{v} \in \mathcal{W}_{\rm red}$ is equivalent to $\mathbf{w} \in \mathcal{W}_{\rm red}$ then there exists a uniquely determined unitary map,

$$
\mathcal{Q}_{\mathbf{v},\mathbf{w}} : \mathcal{H}_{\mathbf{v}} \to \mathcal{H}_{\mathbf{w}} : \xi_{v_1} \otimes \dots \otimes \xi_{v_n} \mapsto \xi_{v_{\sigma(1)}} \otimes \dots \otimes \xi_{v_{\sigma(n)}}, \tag{1.2}
$$

where σ is as in [CaFi14, Lemma 1.3 (4)]. Since every $\mathbf{v} \in \mathcal{W}_{\rm red}$ has a unique minimal form $\mathbf{v}'$ we may simply write $\mathcal{Q}_{\mathbf{v}}$ for $\mathcal{Q}_{\mathbf{v},\mathbf{v}'}$.

Define the *graph product Hilbert space* $(\mathcal{H}, \Omega)$ by:

$$
\mathcal{H} = \mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w} \in \mathcal{W}_{\rm min}} \mathcal{H}_{\mathbf{w}}.
$$

Here Ω is a distinguished unit vector. The graph product state φ_Γ is the vector state given by Ω. For $v \in V\Gamma$, let $\mathcal{W}_v$ be the set of minimal reduced words $\mathbf{w}$ such that the concatenation $v\mathbf{w}$ is still reduced and write $\mathcal{W}_v^c = \mathcal{W}_{\rm min} \setminus \mathcal{W}_v$. Define

$$
\mathcal{H}(v) = \mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w} \in \mathcal{W}_v} \mathcal{H}_{\mathbf{w}}.
$$

We define the isometry $U_v : \mathcal{H}_v \otimes \mathcal{H}(v) \to \mathcal{H}$ in the following way:

$$\begin{array}{rlcl} U_v: & \mathcal{H}_v \otimes \mathcal{H}(v) & \longrightarrow & \mathcal{H} \\ & \xi_v \otimes \Omega & \xrightarrow{\simeq} & \Omega \\ & \mathcal{H}_v^\circ \otimes \Omega & \xrightarrow{\simeq} & \mathcal{H}_v^\circ \\ & \xi_v \otimes \mathcal{H}_{\mathbf{w}} & \xrightarrow{\simeq} & \mathcal{H}_{\mathbf{w}} \\ & \mathcal{H}_v^\circ \otimes \mathcal{H}_{\mathbf{w}} & \xrightarrow{\simeq} & \mathcal{Q}_{v\mathbf{w}}(\mathcal{H}_v^\circ \otimes \mathcal{H}_{\mathbf{w}}) \end{array}$$

Here the actions are understood naturally. Observe that, for any reduced word $\mathbf{w}$, the word $v\mathbf{w}$ is not reduced if and only if $\mathbf{w}$ is equivalent to a reduced word that starts with v. It follows that U_v is surjective, hence unitary. Define, for $v \in V\Gamma$, the faithful unital normal $*$-homomorphism $\lambda_v : \mathcal{B}(\mathcal{H}_v) \to \mathcal{B}(\mathcal{H})$ by

$$\lambda_v(x) = U_v(x \otimes 1)U_v^* \quad \text{for all} \quad x \in \mathcal{B}(\mathcal{H}_v).$$

Observe the λ_v intertwines the vector states ω_{ξ_v} and ω_Ω. The graph product M_Γ is defined as the von Neumann algebra generated by $\cup_{v \in V\Gamma}\lambda_v(\mathsf{M}_v)$. We shall identify M_v as a von Neumann subalgebra of M_Γ and omit λ_v in the notation. If each $\mathsf{M}_v, v \in V\Gamma$ is a II_1-factor, then so is M_Γ, see [CaFi14, Corollary 2.29]. We shall use the fact that M_v and M_w commute whenever $(v, w) \in E\Gamma$ without further reference, see [CaFi14, Section 2]. Finally, we mention that graph products satisfy a universal property for which we refer to [CaFi14, Proposition 2.22].

2. Graph products of free Araki-Woods factors

In this section we consider graph products of free Araki-Woods factors and prove a moment formula. For free products these results have been obtained by Speicher [Spe92].

2.1. **Preliminaries on partitions.** We let $\mathcal{P}(1, \ldots, n)$ be the set of all partitions of the set $\{1, \ldots, n\}$. We let $\mathcal{P}_2(1, \ldots, n)$ be the set of all pair partitions, meaning that every equivalence class consists exactly of 2 elements. In particular the latter set is empty if n is odd. Let $\mathcal{V} \in \mathcal{P}_2(1, \ldots, n)$ and write $\mathcal{V} = \{(e_1, z_1), \ldots, (e_r, z_r)\}, 2r = n, e_i < z_i$. Let $I(\mathcal{V})$ be the set of all pairs (k, l) such that $e_k < e_l < z_k < z_l$. Let Γ be a simplicial graph and let $\mathbf{v}$ be a (not nessecarily reduced) word of length n. We let $\mathcal{P}(\mathbf{v})$ be the set of partitions $\mathcal{V}$ of $(1, \ldots, n)$ with the property that if k and l are equivalent in $\mathcal{V}$ then this implies that $v_k = v_l$. We let $\mathcal{P}_2(\mathbf{v})$ be the subset of $\mathcal{P}(\mathbf{v})$ of pair partitions. For $\mathcal{V} \in \mathcal{P}_2(\mathbf{v})$ we define $I_\Gamma(\mathcal{V})$ as the subset of $I(\mathcal{V})$ consisting of all pairs (k, l) such that $e_k < e_l < z_k < z_l$ and $(v_{e_k}, v_{e_l}) \notin E\Gamma$. A pair partition $\mathcal{V}$ is called admissible if $I(\mathcal{V})$ is empty and it is called Γ-admissible if $I_\Gamma(\mathcal{V})$ is empty. We denote $\mathcal{P}_{2,\mathrm{nc},\Gamma}(\mathbf{v})$ for the subset of $\mathcal{P}_2(\mathbf{v})$ consisting of all Γ-admissible partitions (these can be considered as "non-crossing partitions up to permutations coming from edges of Γ"). Let $\mathcal{V} \in \mathcal{P}(\mathbf{v})$ we say that a sequence of indices $(i_1, \ldots, i_n)$ is of class $\mathcal{V}$ if the following property holds: $i_k = i_l$ and $v_k = v_l$ if and only if k and l are equivalent in $\mathcal{V}$.

2.2. **Free Araki-Woods factors, graph products and a moment formula.** We recall the construction of the free Araki-Woods factors with trivial orthogonal transformation group. See [Shl97] and also [Hia01]. Let $\mathcal{H}$ be $\mathbb{C}^2$ be the two dimensional Hilbert space with orthonormal basis vectors f_1 and f_2. Let $\mathcal{K} = \oplus_{n=1}^\infty \mathcal{H}^{\otimes n} \oplus \mathbb{C}\cdot\Omega$ where Ω is a distinguished (vacuüm) unit vector. For $f \in \mathcal{H}$ let $a^*(f)$ be the creation operator and $a(f)$ be the annihilation operator:

$$\begin{array}{rcll} a^*(f)\xi & = & f \otimes \xi, & \xi \in \mathcal{H}^{\otimes n}, \\ a(f)\Omega & = & 0, & \\ a(f)g \otimes \xi & = & \langle g, f\rangle \xi, & \xi \in \mathcal{H}^{\otimes n}. \end{array} \tag{2.1}$$

Set,

$$g_1 = a^*(f_1) + a(f_1), \qquad g_2 = a^*(f_2) + a(f_2). \tag{2.2}$$

Let P be the von Neumann algebra generated by g_1 and g_2. In fact by [Shl97, Theorem 2.11] P is equal to the free product of the von Neumann algebras generated by g_1 and g_2. Let Γ be a simplicial graph. We equip the notions introduced so far in this subsection with a $v \in V\Gamma$ to distinguish them for different vertices. For every $v \in V\Gamma$ let $\mathsf{P}_v := \mathsf{P}$ and let $\mathcal{K}_v := \mathcal{K}$ be the Hilbert space it acts on. For P_v we denote $g_{1,v}$ and $g_{2,v}$ for

the generators defined in (2.2). We set $a^*_{i,v}$ for $a^*(f_{i,v})$ as in (2.1). Let P_Γ be the graph product von Neumann algebra of the $\mathsf{P}_v, v \in V\Gamma$ with graph product state τ_Γ.

Theorem 2.1. *Let $d_j \in \{g_{i,v} \mid v \in V\Gamma, i \in \{1,2\}\}, 1 \leq j \leq n$. Then,*

- *$\tau_\Gamma(d_1 d_2 \ldots d_n)$ is 0 in case n is odd.*
- *Let v_j be such that $d_j = g_{i,v_j}$ for either $i = 1$ or $i = 2$ and suppose that n is even, say $n = 2r$. Then,*

$$\tau_\Gamma(d_1 d_2 \ldots d_n) = \#\mathcal{P}_{2,\mathrm{nc},\Gamma}(\mathbf{v}).$$

Proof. Let $\mathcal{K}_\Gamma$ be the graph product Hilbert space on which P_Γ acts. Let $\mathcal{W}_0$ be the set of all words that are I-equivalent to the words in $\mathcal{W}_{\min}$. For a word $\mathbf{v} \in \mathcal{W}_0$ we recall that we have set $\mathcal{H}_{\mathbf{v}} = \mathcal{H}^\circ_{v_1} \otimes \ldots \otimes \mathcal{H}^\circ_{v_n}$. We say that the vectors in the latter space have symbol $\mathbf{v}$. We have $\mathcal{K}_\Gamma \simeq \oplus_{\mathbf{v} \in \mathcal{W}_0} \mathcal{H}_{\mathbf{v}}$ by associativity of tensor products. We describe the actions of the creation and annihilation operators:

$$a^*_{i,v_j} \xi_1 \otimes \ldots \otimes \xi_n = \xi_{k_1} \otimes \ldots \otimes \xi_{k_s} \otimes e_{i,v_j} \otimes \xi_{k_{s+1}} \otimes \ldots \otimes \xi_{k_n},$$

where $\xi_1 \otimes \ldots \otimes \xi_n$ has symbol $\mathbf{w} \in \mathcal{W}_0$ and $w_{k_1} \ldots w_{k_s} v_j w_{k_{s+1}} \ldots w_{k_n}$ is the unique word in $\mathcal{W}_0$ that is II-equivalent to $v_j w_1 \ldots w_n$ with $w_{k_s} \neq v_j$ and which satisfies the property that in case $w_{k_i} = w_{k_{i+1}}$ we must have $k_{i+1} = k_i + 1$. Furthermore for a vector $\xi_1 \otimes \ldots \otimes \xi_n$ with symbol $\mathbf{w}$ we find:

$$a_{i,v_j} \xi_1 \otimes \ldots \otimes \xi_n = \begin{cases} \langle \xi_{k_s}, f_i \rangle \xi_{k_1} \otimes \ldots \otimes \widehat{\xi}_{k_s} \otimes \ldots \otimes \xi_{k_n} & \text{in case } (*) \text{ below holds,} \\ 0 & \text{else,} \end{cases}$$

where $(*)$ means that there exist $k_1 \ldots k_n$ determined by the property that $v_j w_{k_1} \ldots \widehat{w}_{k_s} \ldots w_{k_n}$ is II-equivalent to $w_1 \ldots w_n$ (so in particular $w_{k_s} = v_j$), $w_{k_{s-1}} \neq v_j$, $w_{k_1} \ldots \widehat{w}_{k_s} \ldots w_{k_n}$ is in $\mathcal{W}_0$ and in case $w_{k_i} = w_{k_{i+1}}$ we must have $k_{i+1} = k_i + 1$.

The action of $d_1 \ldots d_n = \sum_{(k_1, \ldots, k_n) \in \{1,*\}^n} a^{k_1}_{i_1,v_1} \ldots a^{k_n}_{i_n,v_n}$ on Ω is thus described by sums of n creation/annihilation operators. If the trace,

$$\tau_\Gamma(a^{k_1}_{i_1,v_1} \ldots a^{k_n}_{i_n,v_n}) = \langle a^{k_1}_{i_1,v_1} \ldots a^{k_n}_{i_n,v_n} \Omega, \Omega \rangle,$$

is non-zero, then we must have exactly $\frac{n}{2}$ creation operators and $\frac{n}{2}$ annihilation operators occuring in $a^{k_1}_{i_1,v_1} \ldots a^{k_n}_{i_n,v_n}$ and in particular n must be even. This proves the first statement. So let n be even and let $(k_1, \ldots, k_n) \in \{1,*\}^n$ be such that exactly half of the terms equals 1 and the other half equals $*$. We associate a pair partition to any term $a^{k_1}_{i_1,v_1} \ldots a^{k_n}_{i_n,v_n}$ with non-zero trace in the following way. We connect $s < r$ if $k_s = 1$ and $k_r = *$, $v_s = v_r$ and a_{i_s,v_s} annihilates the vector that was created by $a^*_{i_r,v_r}$. Call the resulting pair partition $\mathcal{V}_{k_1,\ldots,k_n}$.

Claim: Let $\mathcal{S}$ be the set $\left\{ \mathcal{V}_{k_1,\ldots,k_n} \mid (k_1, \ldots, k_n) \in \{1,*\}^n \text{ and } \tau(a^{k_1}_{i_1,v_i} \ldots a^{k_n}_{i_n,v_n}) \neq 0 \right\}$. Then we have $\mathcal{S} = \mathcal{P}_{2,\mathrm{nc},\Gamma}(\mathbf{v})$.

Proof of the Claim: $\subseteq$. Suppose the inclusion does not hold. Then there exist a partition $\mathcal{V} \in \mathcal{S}$ with $e_a < e_b < z_a < z_b$ such that $(e_a, z_a) \in \mathcal{V}, (e_b, z_b) \in \mathcal{V}$ and $(v_a, v_b) \notin E\Gamma$. This means that

$$a^{k_{e_b}}_{i_{e_b},v_{e_b}} (a^{k_{e_b+1}}_{i_{e_b}+1,v_{e_b+1}} \ldots a^{k_n}_{i_n,v_n} \Omega) = 0,$$

which contradicts that $\mathcal{V} \in \mathcal{S}$.

$\supseteq$. Again suppose that the inclusion does not hold. Then there exists a pair partition $\mathcal{V} \in \mathcal{P}_{2,nc,\Gamma}$ such that

$$\tau(a^{k_1}_{i_1,v_1} \ldots a^{k_n}_{i_n,v_n}) = 0, \tag{2.3}$$

where the k_i are the unique indices determined by:

$$k_i = \begin{cases} 1 & \text{if } i = z_a \text{ for some } (e_a, z_a) \in \mathcal{V} \text{ with } e_a < z_a, \\ * & \text{if } i = e_a \text{ for some } (e_a, z_a) \in \mathcal{V} \text{ with } e_a < z_a. \end{cases}$$

In particular, $\mathcal{V} = \mathcal{V}_{k_1,\ldots,k_n}$. But if (2.3) holds and taking into account that $a^{k_1}_{i_1,v_1} \ldots a^{k_n}_{i_n,v_n}$ creates as many vectors as it annihilates (i.e. exactly half of the k_i's equal $*$), this shows that we must have

$$a^{k_{e_b}}_{i_{e_b},v_{e_b}} (a^{k_{e_b+1}}_{i_{e_b}+1,v_{e_b+1}} \ldots a^{k_n}_{i_n,v_n} \Omega) = 0,$$

for some index e_b for which $k_{e_b} = 1$. This can only happen if there were indices e_a, z_a, z_b such that $e_a < e_b < z_a < z_b$ and $(e_a, z_a) \in \mathcal{V}, (e_b, z_b) \in \mathcal{V}$ such that $(v_a, v_b) \notin E\Gamma$. This contradicts that $\mathcal{V} \in \mathcal{P}_{2,nc,\Gamma}(\mathbf{v})$.

Remainder of the proof: We now have, putting $2r = n$,

$$\begin{aligned}\tau_\Gamma(d_1 \ldots d_n) &= \sum_{\mathcal{V}=\{(e_1,z_1),\ldots,(e_r,z_r)\}\in\mathcal{P}_{2,\mathrm{nc},\Gamma}(\mathbf{v})} \langle a^{k_1}_{i,v_1} \ldots a^{k_n}_{i_n,v_n}\Omega, \Omega\rangle \\ &= \sum_{\mathcal{V}=\{(e_1,z_1),\ldots,(e_r,z_r)\}\in\mathcal{P}_{2,\mathrm{nc},\Gamma}(\mathbf{v})} \prod_{k=1}^{r} \langle f_{i_{e_k}}, f_{i_{z_k}}\rangle \\ &= \#\mathcal{P}_{2,\mathrm{nc},\Gamma}(\mathbf{v}).\end{aligned}$$

This concludes the theorem. □

Finally, we recall the following fact from either [Ric05] or [Shl97].

Theorem 2.2. *P is a II_1-factor. In particular $\mathsf{P}_v := \mathsf{P}$ and hence P_Γ is a II_1-factor.*

2.3. **A Speicher type limit theorem.** We prove a version of Speicher's central limit theorem adapted to graph products. See [Spe92] for the original theorem. For a simplicial graph Γ and some index set I we let $s : I \times V\Gamma \times I \times V\Gamma \to \{-1, 1\}$ be a sign function. Let $\mathbf{v}$ be a word with letters in $V\Gamma$ and let $\mathcal{V} \in \mathcal{P}_2(\mathbf{v})$. Let n be the length of the word $\mathbf{v}$. We set,

$$t(\mathcal{V}) = \lim_{N\to\infty} \frac{1}{N^{n/2}} \sum_{\substack{i_1,\ldots,i_n=1 \\ (i_1,\ldots,i_n) \text{ is of class } \mathcal{V}}}^{N} \left(\prod_{(a,b)\in I_\Gamma(\mathcal{V})} s(i_{e_a}, v_{e_a}, i_{e_b}, v_{e_b}) \right).$$

The following theorem can in principle be derived from [Spe92, Theorem 1] (see [Nou06, Remark p.303]) but as our theorem involves the extra condition (4) below and [Nou06] only sketches the argument we give the complete proof.

Theorem 2.3. *Let M be a von Neumann algebra with normal faithful state φ. Let Γ be a simplicial graph. For $v \in V\Gamma$ and $i \in \mathbb{N}$ let $b_{i,v} \in \mathsf{M}$ be a self-adjoint operator. Presume that φ and the $b_{i,v}$'s satisfy the following properties:*

(1) For any set of k different indices $(i_1, v_1), \ldots, (i_k, v_k)$ and powers $l_1, \ldots, l_k$ we have

$$\varphi(b^{l_1}_{i_1,v_1} \ldots b^{l_k}_{i_k,v_k}) = \varphi(b^{l_1}_{i_1,v_1}) \ldots \varphi(b^{l_k}_{i_k,v_k}). \tag{2.4}$$

(2) For all $i \in \mathbb{N}$ and $v \in V\Gamma$, we have mean $\varphi(b_{i,v}) = 0$ and covariance $\varphi(b^2_{i,v}) = 1$.

(3) There is a sign function $s : \mathbb{N}\times V\Gamma\times\mathbb{N}\times V\Gamma \to \{-1, 1\}$ satisfying $s(i, v, j, w) = 1$ whenever $(v, w) \in E\Gamma$ such that:

$$b_{i,v}b_{j,w} = s(i, v, j, w)b_{j,w}b_{i,v}. \tag{2.5}$$

(4) There exists a constant C such that for any choice of indices $i_1, \ldots, i_n$ and vertices $v_1, \ldots, v_n$, we have $|\varphi(b_{i_1,v_1} \ldots b_{i_n,v_n})| \leq C$.

For $v \in V\Gamma$ and $N \in \mathbb{N}$ set

$$S_{N,v} := \frac{b_{1,v} + \ldots + b_{N,v}}{\sqrt{N}}.$$

Now fix a (not necessarily reduced) word $\mathbf{v} = v_1 \ldots v_n$. Assume that the quantity $t(\mathcal{V})$ exists for every $\mathcal{V} \in \mathcal{P}_2(\mathbf{v})$. Then we have,

$$\lim_{N\to\infty} \varphi(S_{N,v_1} \ldots S_{N,v_n}) = \begin{cases} 0 & n \text{ is odd.} \\ \sum_{\mathcal{V}\in\mathcal{P}_2(\mathbf{v}),\mathcal{V}=\{(e_1,z_1),\ldots,(e_r,z_r)\}} t(\mathcal{V}) & n = 2r. \end{cases}$$

Proof. We shall calculate the limit $N \to \infty$ of the quantity:

$$M_N := \varphi(S_{N,v_1} \dots S_{N,v_n}) = \frac{1}{N^{n/2}} \sum_{i_1, \dots, i_n = 1}^{N} \varphi(b_{i_1,v_1} \dots b_{i_n,v_n}).$$

We can split this expression as:

$$M_N = \frac{1}{N^{n/2}} \sum_{\mathcal{V} \in \mathcal{P}(\mathbf{v})} \sum_{\substack{i_1, \dots, i_n = 1 \\ (i_1, \dots, i_n) \text{ is of class } \mathcal{V}}}^{N} \varphi(b_{i_1,v_1} \dots b_{i_n,v_n}).$$

We now compute the limits of the summands for different classes of $\mathcal{V}$.

Step 1. Consider $\mathcal{V} \in \mathcal{P}(\mathbf{v})$ with $\mathcal{V} = \{V_1, \dots, V_p\}$ where at least for one i the class V_i contains only one element. Using the (anti-)commutation relations (2.5) to write $b_{i_1,v_1} \dots b_{i_n,v_n}$ into the form (2.4) and then using that $\varphi(b_{i,v_i}) = 0$, we find that $\varphi(b_{i_1,v_1} \dots b_{i_n,v_n}) = 0$.

Step 2. We shall now assume that $\mathcal{V} = \{V_1, \dots, V_p\}$ and $\#V_i \geq 2$. This implies in particular that $p \leq n/2$. By assumption there exists a constant C such that for all indices $(i_1, \dots, i_n)$ of class $\mathcal{V}$ we have $|\varphi(b_{i_1,v_1} \dots b_{i_n,v_n})| \leq C$. We then estimate:

$$\left| \frac{1}{N^{n/2}} \sum_{\substack{i_1, \dots, i_n = 1 \\ (i_1, \dots, i_n) \text{ is of class } \mathcal{V}}}^{N} \varphi(b_{i_1,v_1} \dots b_{i_n,v_n}) \right| = C \frac{A_{p,\mathbf{v},N}}{N^{n/2}}. \tag{2.6}$$

Here $A_{p,\mathbf{v},N}$ is the number of summands, which equals $N(N-1)\dots(N-p_1+1)N(N-1)\dots(N-p_2+1)\dots N(N-1)\dots(N-p_k+1)$ where the numbers $p_1, \dots, p_k$ are defined as follows. Let $w_1, \dots, w_k$ be the different letters of which $\mathbf{v}$ exists. Then we may write $\mathcal{V} = \mathcal{V}_{w_1} \cup \dots \cup \mathcal{V}_{w_k}$ and $\mathcal{V}_{w_i} = (V_{w_i,1}, \dots, V_{w_i,p_i})$, where $k \in V_{w_i,j}$ implies that $v_k = w_i$. Now taking the limit $N \to \infty$ of (2.6) will give 0 in case $p_1 + \dots + p_k < n/2$. This means that if the limit $N \to \infty$ of (2.6) is non-zero, then each $V_{w_i,j}$ must have exactly 2 elements (in particular, it implies the vanishing of odd partitions).

Step 3. Now we may assume that n is even and that the partition $\mathcal{V}$ is of the form $\{(e_1, z_1), \dots, (e_r, z_r)\}$ with $e_i < z_i$. So $2r = n$. First suppose that $\mathcal{V}$ is Γ-admissible. This implies that there exists an index m such that $v_{e_m} = v_{z_m}$ and for every $e_m < a < z_m$ we have $v_a \in \mathsf{Link}(v_{e_m})$. It follows from the (anti-)commutation relations that b_{e_m} commutes with every b_a for $e_m < a < z_m$ and furthermore $b_{e_m} b_{z_m}$ commutes with every operator b_a for $1 \leq a \leq n$. Therefore, using (2.4) and (anti-)commutation relations, this implies:

$$\begin{aligned}
&\varphi(b_{i_1,v_1} \dots b_{i_n,v_n}) \\
=&\varphi(b_{i_{e_m},v_{e_m}} b_{i_{z_m},v_{z_m}} b_{i_1,v_1} \dots \widehat{b}_{i_{e_m},v_{e_m}} \dots \widehat{b}_{i_{z_m},v_{z_m}} \dots b_{i_n,v_n}) \\
=&\varphi(b_{i_{e_m},v_{e_m}} b_{i_{z_m},v_{z_m}}) \varphi(b_{i_1,v_1} \dots \widehat{b}_{i_{e_m},v_{e_m}} \dots \widehat{b}_{i_{z_m},v_{z_m}} \dots b_{i_n,v_n}) \\
=&\varphi(b_{i_1,v_1} \dots \widehat{b}_{i_{e_m},v_{e_m}} \dots \widehat{b}_{i_{z_m},v_{z_m}} \dots b_{i_n,v_n}).
\end{aligned}$$

Since $\{(e_1, z_1) \dots \widehat{(e_m, z_m)} \dots (e_r, z_r)\}$ is again Γ-admissible we may continue inductively to obtain

$$\varphi(b_{i_1,v_1} \dots b_{i_n,v_n}) = 1.$$

Step 4. Finally, we treat the case that $\mathcal{V} = \{(e_1, z_1), \dots, (e_r, z_r)\}$ is as in the previous step but not necessarily Γ-admissible. In that case we may use the (anti-)commutation relations in order to bring $b_{i_1,v_1} \dots b_{i_n,v_n}$ into

a form that corresponds to a Γ-admissible partition. While doing so one needs exactly the commutation relations,

$$b_{i_{e_a},v_{e_a}} b_{i_{e_b},v_{e_b}} = s(i_{e_a}, v_{e_a}, i_{e_b}, v_{e_b}) b_{i_{e_b},v_{e_b}} b_{i_{e_a},v_{e_a}} \qquad (a,b) \in I_\Gamma(\mathcal{U}).$$

So we get,

$$\begin{aligned} &\frac{1}{N^{n/2}} \sum_{\substack{i_1,\ldots,i_n=1 \\ (i_1,\ldots,i_n) \text{ is of class } \mathcal{V}}}^{N} \varphi(b_{i_1,v_1}\ldots b_{i_l,v_l}) \\ =&\frac{1}{N^{n/2}} \sum_{\substack{i_1,\ldots,i_n=1 \\ (i_1,\ldots,i_n) \text{ is of class } \mathcal{V}}}^{N} \left(\prod_{(a,b)\in I_\Gamma(\mathcal{V})} s(i_{e_a}, v_{e_a}, i_{e_b}, v_{e_b}) \right), \end{aligned}$$

which turns to the quantity as in the statement of the theorem. □

Finally we show that $t(\mathcal{V})$ in the previous theorem can be computed almost everywhere. In order to do so, let $\geq$ be some linear order on $\mathbb{N} \times V\Gamma$. Naturally the symbol $>$ stands for $\geq$ but not equal.

Lemma 2.4. *For Γ a simplicial graph, let $\mathbf{s} = (s(v,i,w,j))_{v,w\in V\Gamma, i,j\in\mathbb{N}}$ be an infinite random matrix with the properties:*

(1) $s(i,v,j,w) = s(j,w,i,v)$ for every $i,j \in I$ and $v,w \in V\Gamma$,
(2) $s(i,v,j,w) = 1$ whenever $(v,w) \in E\Gamma$,
(3) $s(i,v,j,w)$ with $(i,v) > (j,w)$ are independent,
(4) $\mathrm{prob}(s(i,v,j,w) = 1) = p$ and $\mathrm{prob}(s(i,v,j,w) = -1) = q := 1-p$, whenever $(v,w) \notin E\Gamma$.

Let $\mathbf{v}$ be a (not necessarily reduced) word with letters in $V\Gamma$. Then, for almost every $\mathbf{s}$ we have for all $\mathcal{V} \in \mathcal{P}_2(\mathbf{v})$ that

$$t(\mathcal{V}) = (p-q)^{\#I_\Gamma(\mathcal{V})}.$$

Proof. Let $\mathbb{P}$ be a probability measure on the space of all (infinite) random matrices $\mathbf{s}$ satisfying (1) - (4). Let $\mathbb{E}$ be the associated conditional expectation. Let $\mathbf{v}$ have length n and set $n = 2r$. Let $\mathcal{V} = \{(e_1,z_1),\ldots,(e_r,z_r)\}$ be the pair partition in $\mathcal{P}_2(\mathbf{v})$. Define the random variable,

$$X_N = \frac{1}{N^r} \sum_{\substack{i_1,\ldots,i_n=1 \\ (i_1,\ldots,i_n) \text{ is of class } \mathcal{V}}}^{N} \left(\prod_{(a,b)\in I_\Gamma(\mathcal{V})} s(i_{e_a}, v_{e_a}, i_{e_b}, v_{e_b}) \right).$$

The random variables $s(i_{e_a}, v_{e_a}, i_{e_b}, v_{e_b}$ occurring in the product on the right hand side are all independent and have expectation value $p-q$. Hence,

$$\mathbb{E}(X_N) = \frac{1}{N^r} \sum_{\substack{i_1,\ldots,i_n=1 \\ (i_1,\ldots,i_n) \text{ is of class } \mathcal{V}}}^{N} (p-q)^{\#I_\Gamma(\mathcal{V})}.$$

We now wish to conclude that

$$\lim_{N\to\infty} X_N = \lim_{N\to\infty} \mathbb{E}(X_N) = (p-q)^{\#I_\Gamma(\mathcal{V})}, \qquad \text{almost everywhere,}$$

where the first equality remains to be proved and the second equality follows from an elementary combinatorical argument similar to Step 2 in the proof of Theorem 2.3. It suffices to prove that for any $\alpha > 0$ we have

$\lim_{N\to\infty}\mathbb{P}(\{\sup_{M\geq N}|X_M-\mathbb{E}(X_M)|\geq\alpha\})=0$, see [Bau74, Lemma 19.5]. From Tschebycheff's inequality we get,

$$\mathbb{P}(\{\sup_{M\geq N}|X_M-\mathbb{E}(X_M)|\geq\alpha\})\leq\sum_{M=N}^{\infty}\mathbb{P}(\{|X_M-\mathbb{E}(X_M)|\geq\frac{\alpha}{2}\})\leq\frac{4}{\alpha^2}\sum_{M=N}^{\infty}V(X_M), \tag{2.7}$$

where the variance is given by $V(X_M)=\mathbb{E}(X_M^2)-\mathbb{E}(X_M)^2$ is the variance of X_M. Now we have,

$$V(X_M)=\frac{1}{M^{2r}}\sum_{\substack{i_1,\ldots,i_{2r}=1\\(i_1,\ldots,i_{2r})\text{ is of class }\mathcal{V}}}^{M}\;\sum_{\substack{j_1,\ldots,j_{2r}=1\\(j_1,\ldots,j_{2r})\text{ is of class }\mathcal{V}}}^{M}\mathbb{E}\Bigg(\prod_{(a,b)\in I_\Gamma(\mathcal{V})}s(i_{e_a},v_{e_a},i_{e_b},v_{e_b})$$
$$s(j_{e_a},v_{e_a},j_{e_b},v_{e_b})-(p-q)^{2\#I_\Gamma(\mathcal{V})}\Bigg).$$

If at most two of the i-indices $(i_1,\ldots,i_{2r})$ are equal to j-indices in $(j_1,\ldots,j_{2r})$ then all factors in the product $\prod_{(a,b)\in I_\Gamma(\mathcal{V})}s(i_{e_a},v_{e_a},i_{e_b},v_{e_b})s(j_{e_a},v_{e_a},j_{e_b},v_{e_b})$ are independent and hence its conditional expectation is exactly $(p-q)^{2\#I_\Gamma(\mathcal{V})}$. Thus such indices do not contribute to the sum. The number of remaining summands is of order M^{2r-2}, meaning that $V(X_M)\leq C/M^2$ for some constant C. As the sequence $\frac{1}{M^2}$ is summable we see that the limit $N\to\infty$ of (2.7) goes to 0, which concludes the lemma. □

2.4. **Matrix models.** We now prove that the von Neumann algebra P_Γ introduced in Section 2.2 is embeddable. For the moment assume that Γ is a finite graph. Let $N\in\mathbb{N}$ and set $I_N=\{0,\ldots,N\}$. Let $s:I_N\times V\Gamma\times I_N\times V\Gamma\to\{-1,1\}$ be a sign function satisfying the properties:

(1) $s(i,v,j,w)=s(j,w,i,v)$;
(2) $s(i,v,i,v)=-1$.

We let $x_{i,v}$ be algebraic generators of an algebra $\mathcal{A}$ satisfying the following relations:

$$x_{i,v}x_{j,w}-s(i,v,j,w)x_{j,w}x_{i,v}=2\delta_{i,j}\delta_{v,w}.$$

In particular $x_{i,v}^2=1$ and it follows from these (anti-)commutation relations that $\mathcal{A}$ is finite dimensional. Fix a linear order on $I_N\times V\Gamma$. For $A\subseteq I_N\times V\Gamma$ we set,

$$\prod_{(i,v)\in A}x_{i,v},$$

where the product is taken with respect to the linear order. The sets x_A form a basis of $\mathcal{A}$. We equip $\mathcal{A}$ with the $*$-structure given by $x_{i,v}^*=x_{i,v}$. Let φ be the tracial function $\mathcal{A}\to\mathbb{C}$ defined by $\varphi(x_A)=\delta_{A,\emptyset}$. Then $\langle x,y\rangle=\varphi(y^*x)$ defines an inner product and hence a Hilbert space $L^2(\mathcal{A},\varphi)$. We define partial isometries $a_{i,v}^*$ and $a_{i,v}$ by

$$a_{i,v}^*x_A=\begin{cases}x_{i,v}X_A & \text{if }(i,v)\notin A,\\ 0 & \text{if }(i,v)\in A,\end{cases} \tag{2.8}$$

and

$$a_{i,v}x_A=\begin{cases}x_{i,v}X_A & \text{if }(i,v)\in A,\\ 0 & \text{if }(i,v)\notin A,\end{cases} \tag{2.9}$$

Note that (2.8) is the adjoint of (2.9). Then we set

$$b_{i,v}=a_{i,v}^*+a_{i,v}.$$

These operators satisfy the relations

$$b_{i,v}b_{j,w}=s(i,v,j,w)b_{j,w}b_{i,v}.$$

We set for $N\in\mathbb{N}$ even,

$$S_{N,v,1}=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}b_{2i,v},\qquad S_{N,v,2}=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}b_{2i+1,v},$$

Now we assume that $s : I_\infty \times V\Gamma \times I_\infty \times V\Gamma \to \{-1, 1\}$ is an infinite random matrix with entries being independent identically distributed random variables subject to conditions (1) and (2) and such that whenever $(v, w) \notin E\Gamma$ we have $\mathrm{prob}(s(i, v, j, w) = 1) = \mathrm{prob}(s(i, v, j, w) = -1) = \frac{1}{2}$. The following result is now a consequence of Theorem 2.1, Theorem 2.3 and Lemma 2.4.

Theorem 2.5. *Let Γ be a finite simplicial graph and let $V\Gamma = \{v_1, \ldots, v_n\}$. For any $*$-polynomial Q in $2\#V\Gamma$ non-commutating variables we have*

$$\tau\left(Q(g_{1,v_1}, \ldots, g_{1,v_n}, g_{2,v_1}, \ldots, g_{2,v_n})\right) = \lim_{N\to\infty} \tau\left(Q(S_{N,v_1,1}, \ldots, S_{N,v_n,1}, S_{N,v_1,2}, \ldots, S_{N,v_n,2})\right) \tag{2.10}$$

for almost every s.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We wish to prove that P_Γ (see Section 2.2) is Connes embeddable by defining an injective $*$-homomorphism Φ by setting

$$\Phi\left(Q(g_{1,v_1}, \ldots, g_{1,v_n}, g_{2,v_1}, \ldots, g_{2,v_n})\right) = \prod_{N,\mathcal{U}} Q(S_{N,v_1,1}, \ldots, S_{N,v_n,1}, S_{N,v_1,2}, \ldots, S_{N,v_n,2}).$$

However, the entries of the ultra product on the right hand side may not be bounded. Therefore we need bounded cut-offs and at this point the argument is exactly the same as in [Nou06, Section 3]. Meaning, let C be any constant such that $\|g_{1,v_1}\| \leq C$. Let $\chi_{[-C,C]}$ be the characteristic function of the interval $[-C, C]$. We set $\widetilde{S}_{N,v_i,i} := \chi_{[-C,C]}(S_{N,v,i})S_{N,v,i}$. What is proved in [Nou06, Section 3] is that (2.10) also holds if $S_{N,v_i,i}$ is replaced by $\widetilde{S}_{N,v_i,i}$. As a consequnce:

Theorem 2.6. *The graph product free Araki-Woods factor P_Γ is Connes embeddable.*

3. Embeddability of graph products

Throughout this section Γ is a simplicial graph. The following Lemma 3.1 is proved in the first paragraph of the proof of [Jun05, Lemma 7.18].

Lemma 3.1. *Let N and M be von Neumann algebras. Let $\mathcal{E} : \mathsf{N} \to \mathsf{M}$ be a conditional expectation. Let ω be a normal state on M. Let $f \in \mathsf{N}$ be the support of $\omega \circ \mathcal{E}$ and let $e \in \mathsf{M}$ be the support of ω. Then f commutes with every element in $e\mathsf{M}e$.*

Lemma 3.2. *Let $\mathsf{M}_v, v \in V\Gamma$ be von Neumann algebras with normal faithful tracial state τ_v. Suppose that for every $v \in V\Gamma$ there are von Neumann algebras $\mathsf{A}_{v,i}$ with normal faithful tracial states $\tau_{v,i}$ with a trace preserving embedding:*

$$\pi_v : \mathsf{M}_v \to \prod_{i,\mathcal{U}} [\mathsf{A}_{v,i}, \tau_{v,i}],$$

then there exists a trace preserving embedding,

$$\pi_\Gamma : \mathsf{M}_\Gamma \to \prod_{i,\mathcal{U}} [\mathsf{A}_{\Gamma,i}, \tau_{\Gamma,i}].$$

Proof. Let $j_{v,i} : \mathsf{M}_{v,i} \to \mathsf{M}_{\Gamma,i}$ be the natural trace preserving embedding. The predual maps $(j_{v,i})_* : (\mathsf{M}_{\Gamma,i})_* \to (\mathsf{M}_{v,i})_*$ are contractive and hence induce a mapping in the ultraproduct $(j_v)_* : \prod_{i,\mathcal{U}} (\mathsf{M}_{\Gamma,i})_* \to \prod_{i,\mathcal{U}} (\mathsf{M}_{v,i})_*$. We let

$$j_v : \prod_{i,\mathcal{U}} \mathsf{M}_{v,i} \to \prod_{i,\mathcal{U}} \mathsf{M}_{\Gamma,i}$$

be the dual of this mapping. Note that $\prod_{i,\mathcal{U}} \mathsf{M}_{v,i}$ is the dual of $\prod_{i,\mathcal{U}} (\mathsf{M}_{v,i})_*$ via the pairing $\langle (x_i)^\bullet, (\omega_i)^\bullet \rangle = \lim_{i,\mathcal{U}} \omega_i(x_i)$ and therefore explicitly $j_v((x_i)^\bullet) = (j_{v,i}(x_i))^\bullet$.

Let $e_v, v \in V\Gamma$ be the support projection of the ultraproduct state $(\tau_{v,i})^\bullet$. Let f be the support projection of the ultraproduct state $(\tau_{\Gamma,i})^\bullet$. Note that j_v identifies $\prod_{i,\mathcal{U}} \mathsf{M}_{v,i}$ as a subalgebra of $\prod_{i,\mathcal{U}} \mathsf{M}_{\Gamma,i}$ and $(\tau_{\Gamma,i})^\bullet$ restricts to $(\tau_{v,i})^\bullet$ [CaFi14].

Recall that $e_v \prod_{i,\mathcal{U}} \mathsf{M}_{v,i} e_v = \prod_{i,\mathcal{U}} [\mathsf{M}_{v,i}, \tau_{v,i}]$ and similarly $f \prod_{i,\mathcal{U}} \mathsf{M}_{\Gamma,i} f = \prod_{i,\mathcal{U}} [\mathsf{M}_{\Gamma,i}, \tau_{\Gamma,i}]$. Set $\rho_v : \prod_{i,\mathcal{U}} [\mathsf{M}_{v,i}, \tau_{v,i}] \to \prod_{i,\mathcal{U}} [\mathsf{M}_{\Gamma,i}, \tau_{\Gamma,i}]$ by defining $\rho_v(e_v x e_v) = f j_v(e_v x e_v) f$, where $x \in \prod_{i,\mathcal{U}} \mathsf{M}_{v,i}$. By Lemma

3.1 f commutes with the image of j_v, from which we conclude that ρ_v is a $*$-homomorphism. Set $\alpha_v = \rho_v \circ \pi_v$. Note that α_v is faithful: indeed let $0 \leq x \in \mathsf{M}_v$ be non-zero. Then $(\tau_{\Gamma,i})^\bullet(\alpha_v(x)) = \tau_v(x) \neq 0$. Hence $\alpha_v(x) \neq 0$.

We shall now verify the universal property [CaFi14, Proposition 2.22]. Let $(v,w) \in E\Gamma$. For $x = (x_i)^\bullet$ and $y = (y_i)^\bullet$ in respectively $\prod_{i,\mathcal{U}} \mathsf{M}_{v,i}$ and $\prod_{i,\mathcal{U}} \mathsf{M}_{w,i}$ we have since $x_i y_i = y_i x_i$:

$$\begin{aligned}&\rho_v(e_v x e_v)\rho_{v'}(e_{v'} x e_{v'}) = f j_v(e_v x e_v) f j_w(e_w y e_w) f = f j_v(e_v x e_v) j_w(e_w y e_w) f \\ =& f(j_{v,i}(e_{v,i} x_i e_{v,i}))^\bullet (j_{w,i}(e_{w,i} y_i e_{w,i}))^\bullet f = f(j_{w,i}(e_{w,i} y_i e_{w,i}))^\bullet (j_{v,i}(e_{v,i} x_i e_{v,i}))^\bullet f,\end{aligned}$$

and the latter expression equals $\rho_v(e_v x e_v)\rho_{v'}(e_{v'} x e_{v'})$. So the images of ρ_v and ρ_w commute and hence the images of α_v and α_w commute.

Next, let $\mathbf{v} = v_1 \ldots v_n$ be a reduced word. For $1 \leq k \leq n$ let $a_k \in \mathsf{M}_{v_k}^\circ$. Since $e_{v_k}\left(\prod_{i,\mathcal{U}} \mathsf{A}_{v_k,i}\right) e_{v_k}$ equals $\prod_{i,\mathcal{U}}[\mathsf{A}_{v_k,i}, \tau_{v_k,i}]$ we may approximate $\pi_{v_k}(a_k) \in \prod_{i,\mathcal{U}}[\mathsf{A}_{v_k,i}, \tau_{v_k,i}]$ in the strong topology with a bounded net $(a_{k,i,s})^\bullet_{s\in S}$ where $a_{k,i,s} \in \mathsf{A}_{v_k,i}$ (by Kaplansky's density theorem). Since π_{v_k} is trace preserving it follows that $\lim_{s\in S}\lim_{i,\mathcal{U}} \tau_{v_k,i}(a_{k,i,s}) = 0$. Therefore we may replace $a_{k,i,s}$ by $a_{k,i,s}^\circ := a_{k,i,s} - \tau_{v_k,i}(a_{k,i,s}) \in \mathsf{A}_{v_k,i}^\circ$ and still have $(a_{k,i,s}^\circ)^\bullet_{s\in S} \to \pi_{v_k}(a_k)$ strongly. Then,

$$\begin{aligned}&\tau_\Gamma(\alpha_{v_1}(a_1)\ldots\alpha_{v_n}(a_n)) \\ =&\lim_{s\in S}\tau_\Gamma\left(f(j_{v_1,i}(a_{1,i,s}^\circ))^\bullet f \alpha_{v_2}(a_2)\ldots\alpha_{v_n}(a_n)\right) \\ =&\lim_{s_1\in S_1}\ldots\lim_{s_n\in S_n}\tau_\Gamma\left(f(j_{v_1,i}(a_{1,i,s_1}^\circ))^\bullet f\ldots f(j_{v_n,i}(a_{n,i,s_n}^\circ))^\bullet f\right) \\ =&\lim_{s_1\in S_1}\ldots\lim_{s_n\in S_n}\lim_{i,\mathcal{U}}\tau_{\Gamma,i}\left(j_{v_1,i}(a_{1,i,s_1}^\circ)\ldots j_{v_n,i}(a_{n,i,s_n}^\circ)\right),\end{aligned}$$

which equals zero as $\tau_{\Gamma,i}\left(j_{v_1,i}(a_{1,i,s_1}^\circ)\ldots j_{v_n,i}(a_{n,i,s_n}^\circ)\right) = 0$. Hence we may apply [CaFi14, Proposition 2.22] which concludes that there exists a trace preserving embedding $\pi_\Gamma : \mathsf{M}_\Gamma \to \prod_{i,\mathcal{U}}[\mathsf{A}_{\Gamma,i}, \tau_{\Gamma,i}]$. □

Lemma 3.3. *Let* M *be a type* II_1 *factor with normal faithful tracial state* τ*. Consider* $M_n(\mathbb{C})$ *with normalized trace. There exists a trace preserving embedding* $\varphi_n : M_n(\mathbb{C}) \to \mathsf{M}$*.*

Proof. Let $p_1, \ldots, p_n$ be n mutually orthogonal projections in M with $\tau(p_n) = n^{-1}$. Since $\mathcal{M}$ is a type II factor let $u_{i,j}, i \neq j$ be partial isometries with $u_{i,j}u_{i,j}^* = p_i, u_{i,j}^* u_{i,j} = p_j$. Put $u_{i,i} = p_i$. Let $e_{i,j}$ be the matrix units of $M_n(\mathbb{C})$. Then extending $\varphi_n : e_{i,j} \mapsto u_{i,j}$ linearly gives the required mapping. □

Theorem 3.4. *Let* Γ *be a simplicial graph and for every* $v \in V\Gamma$ *let* M_v *be a* II_1 *factor with normal faithful tracial state* τ_v*. The graph product* M_Γ *is Connes embeddable if and only if for every* $v \in V\Gamma, \mathsf{M}_v$ *is Connes embeddable.*

Proof. Assume that Γ is finite. The only if part is trivial as M_v is a subalgebra of M_Γ. Conversely, for every $v \in V\Gamma$ let $\pi_v : \mathsf{M}_v \to \prod_{i,\mathcal{U}}[\mathsf{A}_{v,i}, \tau_{v,i}]$ be an embedding into an ultraproduct of matrix algebras $\mathsf{A}_{v,i}$ with tracial states $\tau_{v,i}$. Lemma 3.2 yields an embedding $\mathsf{M}_\Gamma \to \prod_{i,\mathcal{U}}[\mathsf{A}_{\Gamma,i}, \tau_{\Gamma,i}]$. Let $\mathsf{P}_{v,i}$ be a free Araki-Woods algebra with vacuüm state $\tau_{v,i}$. $\mathsf{P}_{v,i}$ is a II_1-factor by Theorem 2.2. Let $\mathsf{P}_{\Gamma,i}$ be the graph product of these free Araki-Woods factors. By Lemma 3.3 each $\mathsf{A}_{v,i}$ is an expected subalgebra of $\mathsf{P}_{v,i}$ and so $\mathsf{A}_{\Gamma,i}$ is an expected subalgebra of $\mathsf{P}_{\Gamma,i}$. Hence we must prove that $\prod_{i,\mathcal{U}}[\mathsf{P}_{\Gamma,i}, \tau_{\Gamma,i}]$ is Connes embeddable. In turn by [Jun05, Lemma 7.14] it suffices to prove that each $\mathsf{P}_{\Gamma,i}$ is Connes embeddable. The latter is Theorem 2.6. For infinite Γ the result follows from an inductive limit argument, c.f. the proof of [CaFi14, Corollary 2.17]. □

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